Exercise Sheet #3

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P1. (Problem 3.3.)(Borel–Cantelli lemma) If $(A_n)_{n\in\mathbb{N}}$ is a family of measurable subsets of a probability space $(X, \mathcal{B}, \mu)$ and $\sum_{n\in\mathbb{N}} \mu(A_n) < \infty$, then

$$\mu(\{x \in X \mid x \in A_n \text{ for infinitely many } n \in \mathbb{N}\}) = 0.$$

P2. Let $(X, \mathcal{F}, \mu)$ be a measure space and f a measurable function. Prove the Markov-Chebyshev inequality:

$$\forall \alpha > 0, \mu(|f| > \alpha) \leq \frac{1}{\alpha} \int_{|f|>\alpha} |f|\, d\mu \leq \frac{1}{\alpha} \int |f|\, d\mu.$$

P3. Let $(X, \mathcal{F}_X, \mu)$ and $(Y, \mathcal{F}_Y, \nu)$ be probability spaces, and let $T : X \to Y$ be a measurable function. Define $T\mu(A) := \mu(T^{-1}(A))$ for each $A \in \mathcal{F}_Y$. Prove that $\nu = T\mu$ if and only if for all integrable function f:

$$\int_Y f\, d\nu = \int_X f \circ T\, d\mu.$$

P4. (Problem 3.2.) Let $(X, \mathcal{B}, \mu)$ be a probability space. Let $(A_n)_{n\in\mathbb{N}}$ be a family of measurable sets with $a = inf_{n\in\mathbb{N}}\mu\,(A_n) > 0$. We aim to show that there is a set $E \subseteq \mathbb{N}$ such that $\bar{d}(E) := \limsup_{N\to\infty} \frac{|E\cap\{1,\ldots,N\}|}{N} \geq a$, and for any finite set $F \subseteq E, F \neq \emptyset$, one has $\mu\left(\bigcap_{n\in F} A_n\right) > 0$.

(a) Justify that we can assume, without loss of generality, that $\bigcap_{n\in F} A_n \neq \emptyset$ if and only if $\mu\left(\bigcap_{n\in F} A_n\right) > 0$. To do this, it may help to define the countable set

$$\mathcal{F} = \{F \subseteq \mathbb{N} \mid |F| < \infty, \bigcap_{n\in F} A_n \neq \emptyset, \mu\left(\bigcap_{n\in F} A_n\right) = 0\}.$$

(b) Prove that $\int \limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathbf{1}_{A_n}(x)\, d\mu(x) \geq a$.

(c) Define $E = \{n \in \mathbb{N} \mid x \in A_n\}$ for some suitable $x \in X$. Conclude the proof.